Gaussian beam

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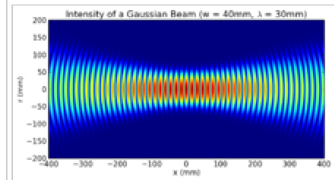
In optics, a **Gaussian beam** is a beam of electromagnetic radiation whose transverse electric field and intensity (irradiance) distributions are well approximated by Gaussian functions. Many lasers emit beams that approximate a Gaussian profile, in which case the laser is said to be operating on the *fundamental transverse mode*, or "TEM_{00} mode" of the laser's optical resonator. When refracted by a diffraction-limited lens, a Gaussian beam is transformed into another Gaussian beam (characterized by a different set of parameters), which explains why it is a convenient, widespread model in laser optics.



Instantaneous intensity of a Gaussian beam.

The mathematical function that describes the Gaussian beam is a solution to the paraxial form of the Helmholtz equation. The solution, in the form of a Gaussian function, represents the complex amplitude of the beam's electric field. The electric field and magnetic field together propagate as an electromagnetic wave. A description of just one of the two fields is sufficient to describe the properties of the beam.

The behavior of the field of a Gaussian beam as it propagates is described by a few parameters such as the spot size, the radius of curvature, and the Gouy phase.[1]

A 5 mW green laser pointer beam profile, showing the TEM_{00} profile

Other solutions to the paraxial form of the Helmholtz equation exist. Solving the equation in Cartesian coordinates leads to a family of solutions known as the Hermite–Gaussian modes, while solving the equation in cylindrical coordinates leads to the Laguerre–Gaussian modes.[2] For both families, the lowest-order solution describes a Gaussian beam, while higher-order solutions describe higher-order transverse modes in an optical resonator.

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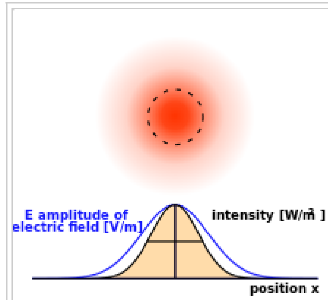


The top portion of the diagram shows the two-dimensional intensity profile of a Gaussian beam that is propagating out of the page. The blue curve, below, is a plot of the electric field amplitude as a function of distance from the center of the beam. The black curve is the corresponding intensity function.

Mathematical form

The Gaussian beam is a transverse electromagnetic (TEM) mode.[3] A mathematical expression for its complex electric field amplitude can be found by solving the paraxial Helmholtz equation, yielding[1]

$$E(r,z) = E_0 \frac{w_0}{w(z)} \exp\left(\frac{-r^2}{w(z)^2} - ikz - ik\frac{r^2}{2R(z)} + i\zeta(z) \right) \ ,$$

where[1]

r is the radial distance from the center axis of the beam,
z is the axial distance from the beam's narrowest point (the "waist"),
i is the imaginary unit (for which $i^2 = -1$),
$k = 2\pi/\lambda$ is the wave number (in radians per meter),
$E_0 = |E(0,0)|$,
$w(z)$ is the radius at which the field amplitude and intensity drop to $1/e$ and $1/e^2$ of their axial values, respectively,
$w_0 = w(0)$ is the waist size,
$R(z)$ is the radius of curvature of the beam's wavefronts, and

$\zeta(z)$ is the Gouy phase shift, an extra contribution to the phase that is seen in Gaussian beams.

Additionally, the field has a time dependence factor $e^{i\omega t}$ that has been suppressed in the above expression.

The corresponding time-averaged intensity (or irradiance) distribution is

$$I(r,z) = \frac{|E(r,z)|^2}{2\eta} = I_0 \left(\frac{w_0}{w(z)} \right)^2 \exp\left(\frac{-2r^2}{w^2(z)} \right) ,$$

where $I_0 = I(0,0)$ is the intensity at the center of the beam at its waist. The constant η is the characteristic impedance of the medium in which the beam is propagating. For free space, $\eta = \eta_0 = \sqrt{\mu_0/\varepsilon_0} = 1/(\varepsilon_0 c) \approx 376.7\ \Omega$.

Beam parameters

The geometry and behavior of a Gaussian beam are governed by a set of **beam parameters**, which are defined in the following sections.

Beam width or spot size

See also: Beam diameter

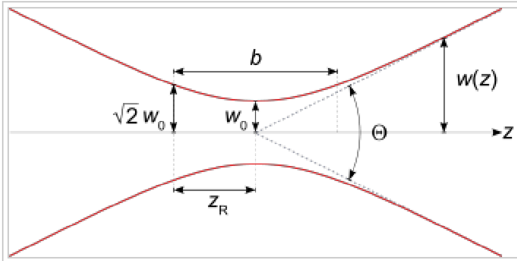
For a Gaussian beam propagating in free space, the spot size (radius) $w(z)$ will be at a minimum value w_0 at one place along the beam axis, known as the *beam waist*. For a beam of wavelength λ at a distance z along the beam from the beam waist, the variation of the spot size is given by[1]

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_\mathrm{R}} \right)^2 } .$$

where the origin of the z-axis is defined, without loss of generality, to coincide with the beam waist, and where[1]

$$z_\mathrm{R} = \frac{\pi w_0^2}{\lambda}$$

is called the Rayleigh range.



Gaussian beam width $w(z)$ as a function of the axial distance z. w_0: beam waist; b: depth of focus; z_R: Rayleigh range; Θ: total angular spread

Rayleigh range and confocal parameter

At a distance from the waist equal to the Rayleigh range z_R, the width w of the beam is[1]

$$w(\pm z_\mathrm{R}) = \sqrt{2} w_0.$$

The distance between these two points is called the *confocal parameter* or *depth of focus* of the beam:

$$b = 2 z_\mathrm{R} = \frac{2 \pi w_0^2}{\lambda} .$$

Radius of curvature

$R(z)$ is the *radius of curvature* of the wavefronts comprising the beam. Its value as a function of position is[1]

$$R(z) = z \left[1 + \left(\frac{z_\mathrm{R}}{z} \right)^2 \right] .$$

Beam divergence

The parameter $w(z)$ increases linearly with z for $z \gg z_\mathrm{R}$. This means that far from the waist, the beam is cone-shaped. The angle between the straight line $r = w(z)$ and the central axis of the beam ($r = 0$) is called the *divergence* of the beam. It is given by[1]

$$\theta \simeq \frac{\lambda}{\pi w_0} \qquad (\theta \text{ in radians}).$$

The total angular spread of the beam far from the waist is then given by

$$\Theta = 2\theta .$$

Because the divergence is inversely proportional to the spot size, a Gaussian beam that is focused to a small spot spreads out rapidly as it propagates away from that spot. To keep a laser beam very well collimated, it must have a large diameter. This relationship between beam width and divergence is due to diffraction. Non-Gaussian beams also exhibit this effect, but a Gaussian beam is a special case where the product of width and divergence is the smallest possible.

Since the gaussian beam model uses the paraxial approximation, it fails when wavefronts are tilted by more than about 30° from the direction of propagation.[4] From the above expression for divergence, this means the Gaussian beam model is valid only for beams with waists larger than about $2\lambda/\pi$.

Laser beam quality is quantified by the beam parameter product (BPP). For a Gaussian beam, the BPP is the product of the beam's divergence and waist size w_0. The BPP of a real beam is obtained by measuring the beam's minimum diameter and far-field divergence, and taking their product. The ratio of the BPP of the real beam to that of an ideal Gaussian beam at the same wavelength is known as M^2 ("M squared"). The M^2 for a Gaussian beam is one. All real laser beams have M^2 values greater than one, although very high quality beams can have values very close to one.

Gouy phase

The *longitudinal phase delay* or *Gouy phase* of the beam is[1]

$$\zeta(z) = \arctan\left(\frac{z}{z_\mathrm{R}}\right) \ .$$

The Gouy phase indicates that as a Gaussian beam passes through a focus, it acquires an additional phase shift of π, in addition to the usual e^{-ikz} phase shift that would be expected from a plane wave.[1]

Complex beam parameter

Main article: Complex beam parameter

Information about the spot size and radius of curvature of a Gaussian beam can be encoded in the complex beam parameter, $q(z)$:[5]

$$q(z) = z + q_0 = z + iz_\mathrm{R} \ .$$

The reciprocal $1/q(z)$ shows the relationship between $q(z)$, $w(z)$, and $R(z)$ explicitly:[5]

$$\frac{1}{q(z)} = \frac{1}{z+iz_\mathrm{R}} = \frac{z}{z^2+z_\mathrm{R}^2} - i\frac{z_\mathrm{R}}{z^2+z_\mathrm{R}^2} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}.$$

The complex beam parameter plays a key role in the analysis of Gaussian beam propagation, and especially in the analysis of optical resonator cavities using ray transfer matrices.

In terms of the complex beam parameter q, a Gaussian field with one transverse dimension is proportional to

$$u(x,z) = \frac{1}{\sqrt{q_x(z)}}\exp\left(-ik\frac{x^2}{2q_x(z)}\right).$$

In two dimensions one can write the potentially elliptical or astigmatic beam as the product

$$u(x,y,z) = u(x,z)\,u(y,z),$$

which for the common case of circular symmetry where $q_x = q_y = q$ and $x^2 + y^2 = r^2$ yields[6]

$$u(r,z) = \frac{1}{q(z)}\exp\left(-ik\frac{r^2}{2q(z)}\right).$$

Power and intensity

Power through an aperture

The power P passing through a circle of radius r in the transverse plane at position z is

$$P(r,z) = P_0\left[1 - e^{-2r^2/w^2(z)}\right] \ ,$$

where

$$P_0 = \frac{1}{2}\pi I_0 w_0^2$$

is the total power transmitted by the beam.

For a circle of radius $r = w(z)$, the fraction of power transmitted through the circle is

$$\frac{P(z)}{P_0} = 1 - e^{-2} \approx 0.865 \ .$$

Similarly, about 95 percent of the beam's power will flow through a circle of radius $r = 1.224 \cdot w(z)$.

Peak and average intensity

The peak intensity at an axial distance z from the beam waist is calculated using L'Hôpital's rule as the limit of the enclosed power within a circle of radius r, divided by the area of the circle πr^2:

$$I(0,z) = \lim_{r\to 0} \frac{P_0 \left[1 - e^{-2r^2/w^2(z)}\right]}{\pi r^2} = \frac{P_0}{\pi} \lim_{r\to 0} \frac{\left[-(-2)(2r)e^{-2r^2/w^2(z)}\right]}{w^2(z)(2r)} = \frac{2P_0}{\pi w^2(z)}.$$

The peak intensity is thus exactly twice the *average intensity*, obtained by dividing the total power by the area within the radius $w(z)$.

Derivation

The Gaussian beam formalism begins with the wave equation for an electromagnetic field in free space or in a homogeneous dielectric medium:[7]

$$\nabla^2 U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2},$$

where U may stand for any one of the six field components E_x, E_y, E_z, B_x, B_y, or B_z. The Gaussian beam formalism proceeds by writing down a solution of the form[7]

$$U(x,y,z,t) = u(x,y,z)e^{-i(kz-\omega t)},$$

where it is assumed that the beam is sufficiently collimated along the z axis that $\partial^2 u/\partial z^2$ may be neglected. Substituting this solution into the wave equation above yields the paraxial approximation to the wave equation:[7]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2ik\frac{\partial u}{\partial z}.$$

Solving this differential equation yields an infinite set of functions, of which the Gaussian beam is the lowest-order solution or *mode*.

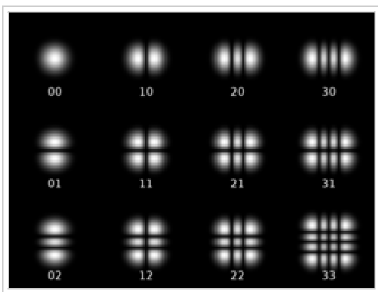
Higher-order modes

See also: Transverse mode

Gaussian beams are just one possible solution to the paraxial wave equation. Various other sets of orthogonal solutions are used for modelling laser beams. In the general case, if a complete basis set of solutions is chosen, any real laser beam can be described as a superposition of solutions from this set. The design of the laser determines which basis set of solutions is most useful. In some cases the output of a laser may closely approximate a single higher-order mode. Hermite-Gaussian modes are particularly common, since many laser systems have Cartesian reflection symmetry in the plane perpendicular to the beam's propagation direction.

Hermite-Gaussian modes

Hermite-Gaussian modes are a convenient description for the output of lasers whose cavity design is not radially symmetric, but rather has a distinction between horizontal and vertical. In terms of the previously defined complex q parameter, the amplitude distribution in the x-plane is proportional to



Twelve Hermite-Gaussian modes

$$u_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n! w_0}\right)^{1/2} \left(\frac{q_0}{q(z)}\right)^{1/2} \left[\frac{q_0}{q_0^*}\frac{q^*(z)}{q(z)}\right]^{n/2} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-i\frac{kx^2}{2q(z)}\right]$$

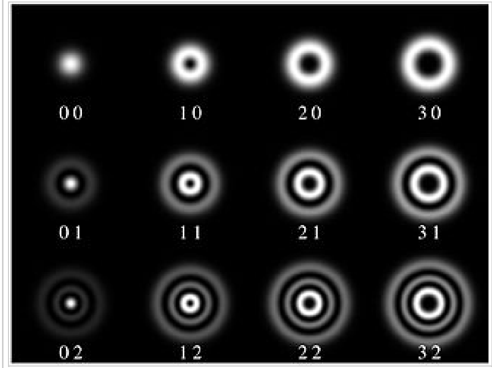
where the function $H_n(x)$ is the Hermite polynomial of order n (physicists' form, i.e. $H_1(x) = 2x$), and the asterisk indicates complex conjugation. For the case $n = 0$ the equation yields a Gaussian transverse distribution.

For two-dimensional rectangular coordinates one constructs a function $u_{mn}(x,y,z) = u_m(x,z)u_n(y,z)$, where $u_n(y,z)$ has the same form as $u_m(x,z)$. Mathematically this property is due to the separation of variables applied to the paraxial Helmholtz equation for Cartesian coordinates.[8]

Hermite-Gaussian modes are typically designated "TEM$_{mn}$", where m and n are the polynomial indices in the x and y directions. A Gaussian beam is thus TEM$_{00}$.

Laguerre-Gaussian modes

If the problem is cylindrically symmetric, the natural solutions of the paraxial wave equation are Laguerre-Gaussian modes. They are written in cylindrical coordinates using Laguerre polynomials



The intensity profiles of twelve Laguerre-Gaussian modes

$$u(r,\phi,z) = \frac{C_{lp}^{LG}}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(ik\frac{r^2}{2R(z)}\right)\exp(il\phi)\exp\left[i(2p+|l|+1)\zeta(z)\right],$$

where L_p^l are the generalized Laguerre polynomials, the radial index $p \geq 0$ and the azimuthal index is l. C_{lp}^{LG} is an appropriate normalization constant; $w(z)$, $R(z)$ and $\zeta(z)$ are beam parameters defined above.

Ince-Gaussian modes

In elliptic coordinates, one can write the higher-order modes using Ince polynomials. The even and odd Ince-Gaussian modes are given by [9]

$$u_\varepsilon\left(\xi,\eta,z\right) = \frac{w_0}{w\left(z\right)}\mathrm{C}_p^m\left(i\xi,\varepsilon\right)\mathrm{C}_p^m\left(\eta,\varepsilon\right)\exp\left[-ik\frac{r^2}{2q\left(z\right)} - \left(p+1\right)\psi_{GS}\left(z\right)\right],$$

where ξ and η are the radial and angular elliptic coordinates defined by

$$x = \sqrt{\varepsilon/2}w\left(z\right)\cosh\xi\cos\eta,$$
$$y = \sqrt{\varepsilon/2}w\left(z\right)\sinh\xi\sin\eta.$$

$C_p^m\left(\eta,\epsilon\right)$ are the even Ince polynomials of order p and degree m, ε is the ellipticity parameter, and $\psi_{GS}\left(z\right) = \arctan\left(z/z_{\mathrm{R}}\right)$ is the Gouy phase. The Hermite-Gaussian and Laguerre-Gaussian modes are a special case of the Ince-Gaussian modes for $\varepsilon = \infty$ and $\varepsilon = 0$ respectively.

Hypergeometric-Gaussian modes

There is another important class of paraxial wave modes in polar coordinates in which the complex amplitude is proportional to a confluent hypergeometric function.

These modes have a singular phase profile and are eigenfunctions of the photon orbital angular momentum. The intensity profile is characterized by a single brilliant ring with a singularity at its center, where the field amplitude vanishes.[10]

$$u_{pm}(\rho,\theta;\zeta) = \sqrt{\frac{2^{p+|m|+1}}{\pi\Gamma(p+|m|+1)}}\frac{\Gamma(1+|m|+\frac{p}{2})}{\Gamma(|m|+1)}i^{|m|+1}\zeta^{\frac{p}{2}}(\zeta+i)^{-(1+|m|+\frac{p}{2})}\rho^{|m|}e^{-\frac{i\rho^2}{(\zeta+i)}}e^{im\phi}{}_1F_1\left(-\frac{p}{2},|m|+1;\frac{r^2}{\zeta(\zeta+i)}\right),$$

where m is integer, $p \geq -|m|$ is real valued, $\Gamma(x)$ is the gamma function and ${}_1F_1(a,b;x)$ is a confluent hypergeometric function.

Some subfamilies of hypergeometric-Gaussian (HyGG) modes can be listed as the modified Bessel-Gaussian modes, the modified exponential Gaussian modes, and the modified Laguerre–Gaussian modes.

The set of hypergeometric-Gaussian modes is overcomplete and is not an orthogonal set of modes. In spite of its complicated field profile, HyGG modes have a very simple profile at the pupil plane:

$$u(\rho,\phi,0) \propto \rho^{p+|m|}e^{-\rho^2+im\phi}.$$

See Optical vortex, which explains that the outcoming wave from a pitch-fork hologram is a sub-family of HyGG modes. The HyGG profile while beam propagates along ζ has a dramatic change and it is not a stable mode below the Rayleigh range.

See also

- Bessel beam
- Tophat beam
- Laser beam profiler

Notes

1. ^ *a b c d e f g h i j* Svelto, pp. 153–5.
2. ^ Siegman, p. 642.
3. ^ Svelto, p. 158.
4. ^ Siegman (1986) p. 630.
5. ^ *a b* Siegman, pp. 638–40.
6. ^ See Siegman (1986) p. 639. Eq. 29
7. ^ *a b c* Svelto, pp. 148–9.
8. ^ Siegman (1986), p645, eq. 54
9. ^ Bandres and Gutierrez-Vega (2004)
10. ^ Karimi et. al (2007)

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